Lecture 16

- Indept of Path

A v.f. is independent of path if the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

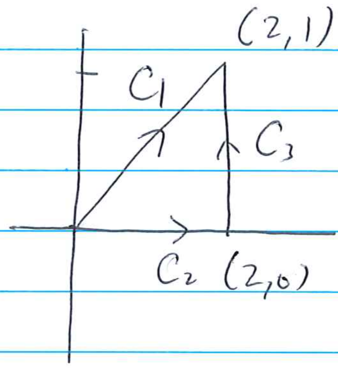
produces the same value from any curve C from points A to B. this is very special.

e.g.

C_1 $\gamma_1(t) = (2t, t)$, $t \in [0,1]$

C_2 $\gamma_2(t) = (2t, 0)$, $t \in [0,1]$

C_3 $\gamma_3(t) = (2, t)$, $t \in [0,1]$



$\vec{F}(x,y) = (y, x^2)$. We've

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 (t, 4t^2) \cdot (2,1)\, dt = 1 + \frac{4}{3} = \frac{7}{3}.$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 (0, 4t^2) \cdot (2,0)\, dt = 0.$$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_0^1 (t, 4) \cdot (0,1)\, dt = 4.$$

then both C_1 and $C_2 + C_3$ go from $A(0,0)$ to $B(2,1)$, but

$$\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2+C_3} \vec{F} \cdot d\vec{r}.$$

It turns out the property of independence of path characterizes gradient vector fields.

<u>Theorem 1</u> $\vec{F}$ is a gradient v.f. $\Rightarrow$ $\vec{F}$ is independent of path.

Some terminologies:

gradient v.f. = conservative v.f. = conservative force

path = piecewise regular curve

Pf: Let Φ be a potential for $\vec{F}$. Take $n=2$ ($n \geq 3$ similar)

$\vec{F} = (P, Q)$, $\nabla \Phi = \vec{F}$ $\Leftrightarrow$

$$\frac{\partial \Phi}{\partial x} = P, \quad \frac{\partial \Phi}{\partial y} = Q.$$

Let A, B be 2 pts and C_1, C_2 two paths from A to B. Need to show

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}.$$

Let $C = C_i$, $i=1,2$ and first assume C is regular.

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b P(\gamma(t))\, x'(t) + Q(\gamma(t))\, y'(t)\, dt$$

where $\gamma : [a,b] \to C$ is a parametrization of C. Then

$$= \int_a^b P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t)\, dt$$

$$= \int_a^b \frac{\partial \Phi}{\partial x}(x(t), y(t))\, x'(t) + \frac{\partial \Phi}{\partial y}(x(t), y(t))\, y'(t)\, dt$$

$$= \int_a^b \frac{d}{dt} \Phi(x(t), y(t))\, dt \qquad \text{(Thanks to Chain Rule)}$$

$$= \Phi(x(t), y(t)) \Big|_a^b$$

$$= \Phi(\gamma(b)) - \Phi(\gamma(a))$$

$$= \Phi(B) - \Phi(A).$$

Hence no matter it is C_1 or C_2,

$$\int_{C_i} \vec{F} \cdot d\vec{r} = \Phi(B) - \Phi(A), \quad i = 1, 2, \qquad (1)$$

and the R.H.S. only depends on the values of Φ at the endpts.

When C is piecewise regular, let's assume, say

~~Remark (1) is a useful formula.~~

$C = C_1 + C_2$ where C_1 runs from A to E, C_2 runs from E to B, are regular (that is, only E is a non-diff. pt)

Applying (1) to C_1, C_2 to get

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1 + C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \Phi(E) - \Phi(A) + \Phi(B) - \Phi(E)$$

$$= \Phi(B) - \Phi(A),$$

(1) still holds. When there are several non-diff. pts, it is clear that (1) is still valid. #

<u>Remark</u> (1) is a useful formula.

Now we prove the converse.

<u>Theorem</u> $\vec{F}$ indept of path $\Rightarrow$ $\vec{F}$ is a gradient v.f.

Proof. Fix a pt A inside the region. For any $B(x,y)$ inside, define

$$\Phi(x,y) = \int_C \vec{F}\cdot d\vec{r}$$

where C is any path from A to $B(x,y)$. It is well-defined because $\vec{F}$ is indept of path.

Now, let C_h be the path $t \mapsto (x+th, y)$, $t \in [0,1]$ where h is a small number. Then $C + C_h$ is a path from A to the pt $(x+h, y)$. So

$$\Phi(x+h,y) - \Phi(x,y) = \int_{C+C_h} \vec{F}\cdot d\vec{r} - \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C \vec{F}\cdot dr + \int_{C_h} \vec{F}\cdot d\vec{r} - \int_C \vec{F}\cdot d\vec{r}$$

$\gamma(t) = (x+th, y)$

$\gamma'(t) = (h, 0)$

$$= \int_{C_r} \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 [P(x+th,y)\,h + Q(x+th,y)\,0]\,dt.$$

$$= h\int_0^1 P(x+th,y)\,dt.$$

$$\therefore \quad \frac{\Phi(x+h,y)-\Phi(x,y)}{h} = \int_0^1 P(x+th,y)\,dt$$

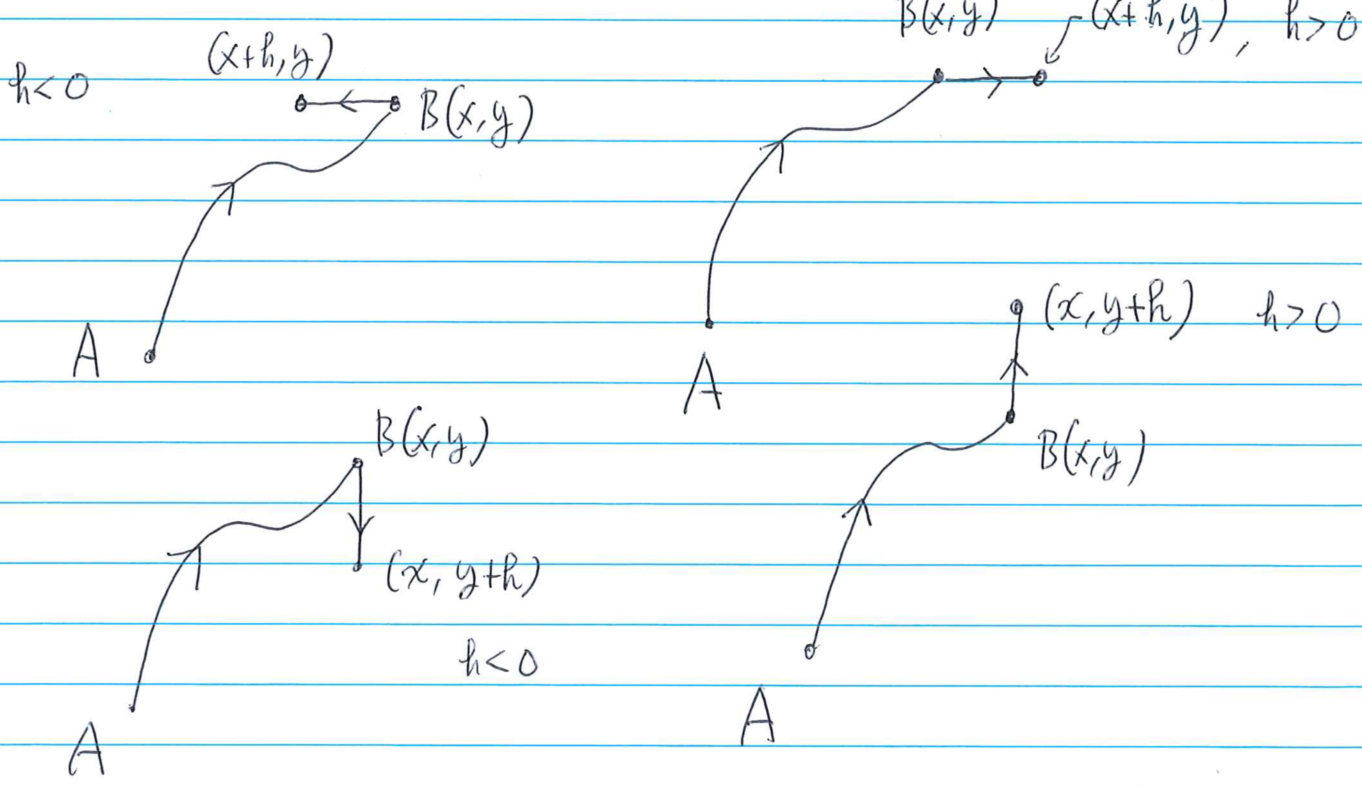
$$\therefore \quad \frac{\partial\Phi}{\partial x}(x,y) = \lim_{h\to 0}\frac{\Phi(x+h,y)-\Phi(x,y)}{h}$$

$$= \lim_{h\to 0}\int_0^1 P(x+th,y)\,dt$$

$$= P(x,y).$$

Similarly, we can show that $\frac{\partial\Phi}{\partial y}$ exists and equals to $Q(x,y)$.

$m \geq 3$ are similarly treated. #



There is an equivalent formulation for indept of path. Let's denote by

$$\oint_C \vec{F}\cdot d\vec{r}$$

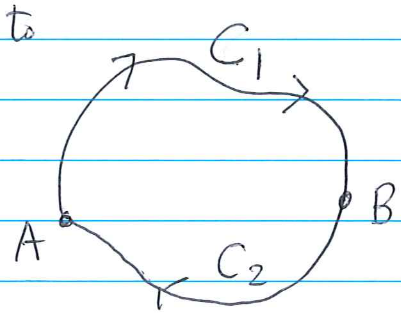
when C is a closed loop, that is, a curve without endpts.

<u>Theorem</u> A v.f. $\vec{F}$ is indpt of path if and only if

$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

for every closed loop C.

Pf: $\Rightarrow$) Pick 2 pts A, B on C and denote C_1 the curve from A to B and C_2 the curve from B to A so that $C = C_1 + C_2$. Now, $-C_2$ is a curve from A to B.



By assumption,

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{-C_2} \vec{F}\cdot d\vec{r}.$$

Using $\displaystyle\int_{-C_2} \vec{F}\cdot d\vec{r} = -\int_{C_2} \vec{F}\cdot d\vec{r}$, we get

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = -\int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = 0.$$

$\Leftarrow$) Let C_1, C_2 two paths from A to B. Then $C = C_1 - C_2$ is a closed loop. So

$$0 = \oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r}$$

$$= \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot dr, \text{ ie}$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}. \quad \#$$